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An Electric Charge System Moving Radially with the Velocity of Light

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Abstract

A solution is found for Maxwell's equations which are associated with a radial flow of electric charge moving with the speed of light.

1. *Introduction*

Recently, Bonnor (1969) discussed the motion of a charge system moving with the velocity of light parallel to a fixed axis. It is the purpose of this paper to show that solutions of Maxwell's equations exist which involve the flow of charge radially out (or in) from a fixed point. Before proceeding to the analysis, it is convenient to recapitulate certain properties of the fields associated with such a charge current distribution. The force per unit volume on a charge density ρ per unit volume moving with speed c in the direction associated with unit vector $\hat{\mathbf{u}}$ is given by, with the usual notation,

$$\rho(\mathbf{E} + c\,\hat{\mathbf{u}} \times \mathbf{B})$$

(the Giorgi system of units is used). If this is zero, and $\hat{\mathbf{u}}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$, it follows that $\rho c\hat{\mathbf{u}}.\mathbf{E}$ vanishes, and so the four-dimensional force density (Sommerfeld, 1964a) vanishes, and also that $\mathbf{E}.\mathbf{B}$, and $\epsilon_0 E^2 - (B^2/\mu_0)$, the two vector invariants of the electromagnetic field (Sommerfeld, 1964b) vanish. The energy density is given by

$$W = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \epsilon_0 E^2 = \frac{B^2}{\mu_0} \tag{1.1}$$

and the Poynting vector is given by

$$\mathbf{E} \times \mathbf{H} = \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)} E^2\,\hat{\mathbf{u}} = Wc\hat{\mathbf{u}} \tag{1.2}$$

The stress tensor (Sommerfeld, 1964c) may be obtained as follows. There is a tension of magnitude $\tfrac{1}{2}\epsilon_0 E^2$ in the direction of $\mathbf{E}$ and a compression of the same magnitude in the directions perpendicular thereto.

Similarly, there is a tension of magnitude $\frac{1}{2}(B^2/\mu_0)$ in the direction of **B** and a compression of the same magnitude in the directions perpendicular thereto. Now

$$\tfrac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\frac{B^2}{\mu_0} = \tfrac{1}{2}W \tag{1.3}$$

and it follows that the stress tensor associated with **E** and **B** is equivalent to a tension of magnitude W in the direction which is perpendicular to both **E** and **B**, that is $\hat{\mathbf{u}}$. There is no stress in the directions perpendicular to $\hat{\mathbf{u}}$.

2. *Radial Solution of Maxwell's Equations*

Any solution of Maxwell's equations can be written in the form (Stratton, 1941)

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{2.1}$$

If the flow of charge is radial, then,

$$\mathbf{E} + c\hat{\mathbf{r}} \times \mathbf{B} = 0 \tag{2.2}$$

It follows that E_r vanishes and hence A_r. Thus, in spherical polar coordinates

$$\mathbf{A} = A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}} \tag{2.3}$$

$$\mathbf{E} = -\frac{\partial A_\theta}{\partial t}\hat{\boldsymbol{\theta}} - \frac{\partial A_\phi}{\partial t}\hat{\boldsymbol{\phi}} \tag{2.4}$$

$$\mathbf{B} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial}{\partial_r}(rA_\theta)\,\hat{\boldsymbol{\phi}} \tag{2.5}$$

Now in order that B_r vanish, it follows that

$$\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = \frac{\theta A_\theta}{\partial\phi}$$

whence

$$A_\theta = \frac{\partial u}{\partial\theta} \quad A_\phi = \frac{1}{\sin\theta}\frac{\partial u}{\partial\phi} \tag{2.6}$$

u is a function of r, θ, ϕ, t. Thus

$$\mathbf{E} = -\frac{\partial^2 u}{\partial\theta\,\partial t}\hat{\boldsymbol{\theta}} - \frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\phi\,\partial t}\hat{\boldsymbol{\phi}} \tag{2.7a}$$

$$\mathbf{B} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\phi}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \tag{2.7b}$$

Substituting equations (2.7a) and (2.7b) in equation (2.2), it follows that

$$0=\left[\frac{\partial^2 u}{\partial\theta\,\partial t}+\frac{c}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\theta}\right)\right]\hat{\boldsymbol{\theta}}-\frac{1}{\sin\theta}\left[\frac{\partial^2 u}{\partial\phi\,\partial t}+\frac{c}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\phi}\right)\right]\hat{\boldsymbol{\phi}} \tag{2.8}$$

This is satisfied if

$$\frac{1}{c}\frac{\partial u}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ru)=0 \tag{2.9}$$

that is

$$u=\frac{1}{r}f\left(\theta,\phi,t-\frac{r}{c}\right) \tag{2.10}$$

f is arbitrary.

The charge and current density are given by

$$\begin{aligned}\mathbf{J}&=\boldsymbol{\nabla}\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\\
&=\frac{1}{\mu_0}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\theta}\right)\right)+\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi^2\,\partial r}(ru)\right]\hat{\mathbf{r}}\right.\\
&\quad\left.-\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\frac{\partial u}{\partial\theta}\right)\hat{\boldsymbol{\theta}}-\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r^2\,\partial\phi}(ru)\,\hat{\boldsymbol{\phi}}\right\}-\epsilon_0\left\{\frac{\partial^2 u}{\partial\theta\,\partial t^2}\hat{\boldsymbol{\theta}}+\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\phi\,\partial t^2}\hat{\boldsymbol{\phi}}\right\}\end{aligned}$$

$$\begin{aligned}J_\theta&=\frac{1}{\mu_0}\left\{-\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\frac{\partial u}{\partial\theta}\right)+\frac{1}{c^2}\frac{\partial^2 u}{\partial\theta\,\partial t^2}\right\}\\
&=\frac{1}{\mu_0 r}\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right\}\frac{\partial u}{\partial\theta}\\
&=0\end{aligned}$$

by virtue of equation (2.10).

Similarly, J_ϕ vanishes, and so

$$\mathbf{J}=J_r\hat{\mathbf{r}}$$

where

$$J_r=\frac{1}{\mu_0}\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial\theta}\right)\right)+\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi^2\,\partial r}(ru)\right]\right\} \tag{2.11}$$

The charge density is given by

$$\begin{aligned}\rho&=\boldsymbol{\nabla}.\mathbf{D}=\epsilon_0\boldsymbol{\nabla}.\mathbf{E}\\
&=\frac{\epsilon_0}{r\sin\theta}\left\{-\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial^2 u}{\partial\theta\,\partial t}\right)-\frac{1}{\sin\theta}\frac{\partial^3 u}{\partial\phi^2\,\partial t}\right\}\end{aligned} \tag{2.12}$$

Substituting the relation (2.9), it follows that

$$J_r=c\rho \tag{2.13}$$

that is, the velocity of flow is $c\hat{\mathbf{r}}$, which is in agreement with the original hypothesis, and so it follows that a consistent field system has been obtained.

It may be remarked that (2.9) is not the only relation which causes equation (2.8) to be satisfied. Any relation of the form

$$\frac{1}{c}\frac{\partial u}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ru)=F(r,t)$$

where F is arbitrary, would in fact do.

However, the fields E and B involve differentiations with respect to θ and ϕ, and consequently F does not matter. It may thus be assumed zero.

It may be observed further that if the sign of c be reversed everywhere, there is a radial inward flow with speed c.

3. *Calculation of Associated Quantities*

The charge density, using equations (2.10) and (2.12), is given by

$$\rho=\frac{-\epsilon_0}{r^2}T\frac{\partial f}{\partial t} \tag{3.1}$$

where T is the operator

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.2}$$

The total charge within a sphere of radius a, centre the origin is therefore

$$Q(a)=\int_0^a r^2\,dr\int_{-\pi}^{\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\,\rho(r)$$

$$=-\epsilon_0\int_0^a dr\int_{-\pi}^{\pi}d\phi\int_0^{\pi}d\theta\sin\theta T\frac{\partial f}{\partial t}\left(\theta,\phi,t\frac{r}{c}\right) \tag{3.3}$$

This may in fact be shown to be zero.

For writing f in terms of surface harmonics (Stratton, 1941b)

$$f(\theta,\phi,t)=\sum_{n=0}^{\infty}\left[a_{no}P_n(\cos\theta)+\sum_{m=1}^{n}(a_{nm}\cos m\phi+b_{nm}\sin m\phi)P_n{}^m(\cos\theta)\right] \tag{3.4}$$

where the a_{nm} and b_{nm} are functions of t.

Now, by the properties of the Legendre functions

$$TP_n{}^m(\cos\phi)\begin{matrix}\cos\\ \sin\end{matrix}m\phi=-n(n+1)P_n{}^m(\cos\phi)\begin{matrix}\cos\\ \sin\end{matrix}m\phi$$

and so

$$Q(a)=\epsilon_0\int_0^a dr\int_{-\pi}^{\pi}d\theta\int_0^{\pi}d\theta\sin\theta\left\{\sum_{n=0}^{\infty}n(n+1)\left[a'_{no}\left(t-\frac{r}{c}\right)P_n(\cos\theta)\right]\right.$$

$$+\sum_{m=1}^{n}\left\{a'_{nm}\left(t-\frac{r}{c}\right)\cos m\phi + b_{nm}\left(t-\frac{r}{c}\right)\sin m\phi\right\}P_n^m\cos\theta\Bigg]$$

$$=2\pi\epsilon_0\int_0^a dr\int_0^\pi d\theta\sin\theta\sum_{n=1}^{\infty}a'_{no}\left(t-\frac{r}{c}\right)n(n+1)P_n(\cos\theta)$$

on integrating with respect to ϕ and dropping the terms in $n=0$. Using the fact that $P_0(\cos\theta)=1$ and the orthogonality properties of Legendre polynomials, it follows that this is zero.

Similarly, the net rate of flow of charge out of the sphere

$$I(a)=\int_{-\pi}^{\pi}d\phi\int_0^\pi\sin\theta\, d\theta[J_r]_{r=a}\,.\,a^2 \tag{3.5}$$

$$=-\epsilon_0 c\int_{-\pi}^{\pi}d\theta\int_0^\pi\sin\theta\, d\theta T\left[\frac{\partial f}{\partial t}\right]_{r=a} \tag{3.6}$$

also vanishes.

This means that the charge distribution is such that the total charge in any spherical shell with centre at the origin is zero, and it cannot therefore be one signed. The density of electromagnetic energy is given by

$$W=\epsilon_0 E^2=\epsilon_0\left[\left(\frac{\partial^2 u}{\partial\theta\,\partial t}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\phi\,\partial t}\right)^2\right]$$
$$=\frac{\epsilon_0}{r^2}\left[\left(\frac{\partial^3 f}{\partial\theta\,\partial t^2}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial^3 f}{\partial\phi\,\partial t^2}\right)^2\right]$$

This is intrinsically positive.

Let $\partial^2 f/\partial t^2=\Gamma$, for convenience

$$W=\frac{\epsilon_0}{r^2}\left[\left(\frac{\partial\Gamma}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial\Gamma}{\partial\theta}\right)^2\right] \tag{3.7}$$

The energy enclosed within a sphere of radius a is

$$\int_0^a r^2 dr\int_{-\pi}^{\pi}d\phi\int_0^\pi\sin\theta\, d\theta W(r,\theta,\phi)$$

$$=\epsilon_0\int_0^a dr\int_{-\pi}^{\pi}d\phi\int^{\pi}\sin\theta\, d\theta\left[\left(\frac{\partial\Gamma}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial\Gamma}{\partial\phi}\right)^2\right] \tag{3.8}$$

The Poynting vector is $Wc\hat{\mathbf{r}}$ and the rate of flow of energy out of the sphere is given by

$$a^2\int_{-\pi}^{\pi}d\phi\int_0^\pi d\theta\sin\theta\frac{\epsilon_0 c}{a^2}\left[\left(\frac{\partial\Gamma}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial\Gamma}{\partial\phi}\right)^2\right]_{r=a} \tag{3.9}$$

which is intrinsically positive, as W is also.

Consider now the integral

$$F=\int_{-\pi}^{\pi} d\phi \int_{0}^{\pi} d\theta \sin\theta\left[\left(\frac{\partial \Gamma}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial \Gamma}{\partial \phi}\right)^2\right] \tag{3.10}$$

$$\int_{-\pi}^{\pi} d\phi\left(\frac{\partial \Gamma}{\partial \phi}\right)^2 = {}_{\pi}^{\pi}\left[\Gamma\frac{\partial \Gamma}{\partial \phi}\right]-\int_{-\pi}^{\pi}\Gamma\frac{\partial^2 \Gamma}{\partial \phi^2}d\phi \tag{3.11a}$$

Γ is, by its nature, periodic in ϕ with period 2π, and so the integrated portion on the Right Hand side of (3.11a) vanishes. Similarly,

$$\int_{0}^{\pi} d\theta \sin\theta\left(\frac{\partial \Gamma}{\partial \theta}\right)^2 = {}_{0}^{\pi}\left[\Gamma\sin\theta\frac{\partial \Gamma}{\partial \theta}\right]-\int_{0}^{\pi}\Gamma\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Gamma}{\partial \theta}\right)d\theta \tag{3.11b}$$

The integrated portion of the right-hand side of (3.11b) vanishes as $\sin\theta$ vanishes at the limits. It follows, therefore, that

$$F=-\int_{-\pi}^{\pi} d\phi \int_{0}^{\pi} d\theta \sin\theta\, \Gamma\left\{\frac{1}{\sin\theta}\frac{\partial \Gamma}{\partial \theta}\left(\sin\theta\frac{\partial \Gamma}{\partial \theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 \Gamma}{\partial \phi^2}\right\}$$

$$=-\int_{-\pi}^{\pi} d\phi \int_{0}^{\pi} d\theta \sin\theta\, \Gamma T \Gamma \tag{3.12}$$

Let

$$\Gamma=\sum_{n=0}^{\infty}\frac{\alpha_{no}}{\sqrt{2}}P_n(\cos\theta)+\sum_{m=1}^{n}(\alpha_{nm}\cos m\phi+\beta_{nm}\sin m\phi)P_n{}^m(\cos\theta)$$

Then if $\Gamma=\Gamma(\theta,\phi,t)$, $\alpha_{nm}=\alpha_{nm}(t)$, $\beta_{nm}=\beta_{nm}(t)$.

Then

$$T\Gamma=-\sum_{n=0}^{\infty}n(n+1)\left\{\frac{\alpha_{no}}{\sqrt{2}}P_n(\cos\theta)+\sum_{m=1}^{n}(\alpha_{nm}\cos m\phi+\beta_{nm}\sin m\phi)P_n{}^m(\cos\theta)\right\}$$

and

$$F=\int_{-\pi}^{\pi} d\theta \int_{0}^{\pi} \sin\theta\, d\theta \sum_{p=0}^{\infty}\left[P_p(\cos\theta)+\sum_{q=1}^{p}(\alpha_{pq}\cos q\phi+\beta_{pq}\cos)P_p{}^q(\cos\theta)\right]$$

$$\times\sum_{n=0}^{\infty}n(n+1)\left[\frac{\alpha_{no}}{\sqrt{2}}(\cos\theta)\right.$$

$$\left.+\sum_{m=0}^{n}(\alpha_{nm}\cos m\phi+\beta_{nm}\cos m\phi)P_n{}^m(\cos\theta)\right]$$

Using the orthogonality relation steps for the Legendre Functions (Stratton, 1941b) and the integral relationships for $\sin m\phi$ and $\cos m\phi$

$$F = 2\pi \sum_{n=1}^{\infty} \left\{ \frac{[n(n+1)]}{2n+1} \alpha_{no}^2 + \sum_{m=1}^{n} \frac{(n)(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} (\alpha_{nm}^2 + \beta_{nm}^2) \right\} \quad (3.13)$$
$$= F(t)$$

if $\alpha_{nm} = \alpha_{nm}(t)$, $\beta_{nm} = \beta_{nm}(t)$.

If the series converges, F is finite. Thus it follows from equation (3.8) that the energy enclosed within a sphere of radius a is

$$G(a) = \epsilon_0 \int_0^a dr\, F\left(t - \frac{r}{c}\right) \quad (3.14)$$

and from equation (3.9) that the outflow of energy from the sphere of radius a is

$$L(a) = \epsilon_0\, cF\left(t - \frac{a}{c}\right) \quad (3.15)$$

Now

$$\frac{dG(a)}{dt} = \epsilon_0 \int_0^a dr\, F'\left(t - \frac{r}{c}\right) = \epsilon_0\, c\left[F(t) - F\left(t - \frac{a}{c}\right)\right]$$
$$= \epsilon_0\, cF(t) - L(a) \quad (3.16)$$

Making a tend to zero, it follows that there is an energy source at the origin working at the rate $\epsilon_0 cF(t)$ which is positive.

Thus a system has been set up involving a radial flow of charge with the speed of light, and which satisfies Maxwell's equations. The net charge within any spherical shell whose centre is that of the flow is zero, and the energy within any sphere of finite radius can be finite. There is, however, a source of energy at the centre.

A corresponding system exists with an inward charge flow. In this case there is a sink of energy at the origin.

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